Ocean Modeling with High Order Finite Element Methods
Benefits of Unstructured Grids

- Geometric fidelity
- Variable resolution across basin
- Focus on dynamical features of interests
- Efficient use of computational resources
- Natural nesting of grids
- Dynamic adaptivity
Spectral Element Ocean Model

<table>
<thead>
<tr>
<th>Advantages</th>
<th>Disadvantages</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$-$p$ FEM</td>
<td>High CPU cost/DOF</td>
</tr>
<tr>
<td>Low numerical dissipation</td>
<td>Low numerical dissipation</td>
</tr>
<tr>
<td>High phase accuracy</td>
<td>Gibbs oscillations</td>
</tr>
<tr>
<td>Good parallel scalability</td>
<td></td>
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<tr>
<td>Geometric Flexibility</td>
<td></td>
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</tbody>
</table>
Example Formulation: Advection-Diffusion

1. **PDE:**  
   \[ q_t + \vec{u} \cdot \nabla q = \nabla \cdot \nu \nabla q \]

2. **Variational form:**
   \[ \int_V \Phi q_t dV = -\int_V (\Phi \vec{u} \cdot \nabla q + \nabla \Phi \cdot \nu \nabla q) dV + \int_{\partial V} \vec{n} \Phi \nu \nabla q dS, \forall \Phi \in H^1_0 \]

3. **Galerkin discretization:** set \( \Phi = h_j \),
   Integrals with Gauss Lobatto quadrature
   Diagonal mass matrix

4. **Time Step (AB3):**
   \[ q^{k+1} = q^k + \Delta t M^{-1} \left( \frac{5f^k - 16f^{k-1} + 23f^{k-2}}{12} \right) \]
Spectral Element in a Nutshell

\[ \xi_1 \xi_2 \ldots \xi_N \xi_{N+1} \]

- Lagrange Polynomial Interpolation

\[ u(\xi) = \sum_{i=1}^{N+1} u_i h_i(\xi), \quad h_i(\xi_j) = \delta_{ij} \]

- Legendre Gauss-Lobatto interpolation and quadrature

\[ (1 - \xi_j^2)L_N(\xi_j) = 0, \quad (\Delta \xi)_{\text{min}} \sim 1/N^2 \]

\[ \int_{-1}^{1} h_i(\xi) h_j(\xi) d\xi \approx \sum_{p=1}^{N+1} h_i(\xi_p) h_j(\xi_p) \omega_i = \delta_{ij} \omega_i \]
Gauss-Lobatto Lagrange Interpolation

linear, quadratic, cubic, and fifth.
Inviscid Smooth Solution $e^{-16x^2}$

Graph showing $T(x, t=10,000 \text{ revolutions})$

- Black line: $SE (K,N)=(4,10)$
- Red line: $CD2 \Delta x=1/100$
SE in 2D

- \( u(\xi, \eta) = \sum_{j=0}^{N} \sum_{i=0}^{N} u_{ij} h_i(\xi) h_j(\eta) \)

- Typical Derivative Calculation: \( O(KN^3) \)

\[
u_{\xi}(\xi_m, \eta_n) = \sum_{i=0}^{N} u_{in} h_i'(\xi_i)\]
2D Numerical Tests
Advection of a Gaussian Hill in a rotating flow field

\[ \vec{u} = 2\pi \left( \frac{1}{2} - y, x - \frac{1}{2} \right), T = e^{-\frac{r^2}{l^2}}, l = \frac{1}{16} \]
Gaussian Hill

Contour lines of the Gaussian hill after 1 rotation
Interpolation order 5
Gaussian Hill-Convergence

Convergence curves for CGM (left) and DGM (right). The labels indicate the number of elements in each direction. The abcissa represent the spectral truncation.
Parallel Scalability

- Serial Computational Time for $KP$ elements: $T_S = PKN^3$

- Computational Time for $K$ elements/processor: $T_p = KN^3$

- Interprocessor Communication $\sim$ to subdomain perimeter:
  \[ T_C \sim 2(K_x + K_y)N \]

- Parallel Speed up:
  \[ S = \frac{T_S}{T_p + T_c} = \frac{P}{1 + \alpha \frac{K_x + K_y}{KN^2}} = \begin{cases} \frac{P}{1 + \alpha K N^{-2}} & \text{square subdomain} \\ \frac{P}{1 + \alpha N^{-2}} & \text{elongated subdomain} \end{cases} \]
Model Suite

- **Two-dimensional**
  - Shallow water equations:
    - SEOM2D (CGM) + Tangent Linear Adjoint
    - SEOM2D (mixed CGM/DGM)
    - Spectral Finite Volume (Outcropping with FCT)
  - Nonhydrostatic Navier Stokes equations (recent)

- **Three-dimensional Hydrostatic**
  - Terrain-Following
    - SEOM3D (CGM)
    - SEOM3D (mixed CGM/DGM)
  - Isopycnal coordinate in the vertical
Abyssal Circulation in Central Indian Ocean

The abyssal Indian Ocean is part of the upwelling limb of the global thermo-haline circulation. Cold and dense Circumpolar Deep Waters (CDW) imported from the south are converted into relatively warm and less dense North Indian Deep Water (NIDW) and exported southward. A shallow system of ridges divides the deep Indian Ocean into smaller basins. The water upwelling in these basins is supplied by CDW inflow through successive deep basins from south to north in three main current systems along the western flanks of the Madagascar Island, the Central Indian and the Ninetyeast Ridges. The circulation pathways of the upwelled water in the 2000-3500 m depth range is relatively unknown. The WOCE data suggest a very circuitious pathway for waters in this depth range with deep saddles on the ridges and fractures through them allowing inter-basin exchange of water.
$^3\text{He}$ along $\sigma_\theta = 37.6$
Dynamical Equations: \( 1\frac{1}{2} \) SWE

\[
\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \mathbf{f} \times \vec{u} + g' \nabla \zeta = \frac{\nabla \cdot (h \nu \nabla \vec{u})}{h}
\]

\[
h_t + \nabla \cdot (h \vec{u}) = Q
\]

\( h = H + \zeta \): layer thickness,
\( \vec{u} \): layer velocity,
\( \mathbf{f} \): Coriolis parameter,
\( g' \): reduced gravity,
\( \nu \): viscosity,
\( Q \): Area mass sink.
Forcing

1. Lateral mass source $q$:
\[
\int_A h_t \psi dA - \int_A h \vec{u} \cdot \nabla \psi dA = \int_A Q \psi dA - \int_\Gamma q \psi dS.
\]

2. Area mass sink: $Q$
\[
Q = \frac{\int_\Gamma q dS}{\int_A dA}
\]

3. No slip on closed boundaries
Computational Grid

Elemental Partition
7 Grid points/element

Average Grid Spacing in km.
Coastline: 2500m isobath
## Numerical Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H$</td>
<td>1000m</td>
</tr>
<tr>
<td>$g' = 0.009$ m²/s</td>
<td>0.009 m²/s</td>
</tr>
<tr>
<td>$c = \sqrt{gh}$</td>
<td>9 m/s</td>
</tr>
<tr>
<td>Rossby radius</td>
<td>$\approx 60$ km</td>
</tr>
<tr>
<td>$q$</td>
<td>2 Sv</td>
</tr>
<tr>
<td>$Q = 1.14 \times 10^{-7}$ m/s</td>
<td>1.14 x 10^{-7} m/s</td>
</tr>
<tr>
<td>$\nu$</td>
<td>40 m²/s</td>
</tr>
<tr>
<td>Munk layer thickness</td>
<td>$\approx 12$ km</td>
</tr>
</tbody>
</table>
Interface Height

Interface Displacement, year = 81.00
max = 0.1810K+02
min = -0.3521K+02
contour from -50 to 30 by 2
Flow Field

Speed (nonlinear color bar)  Velocity Vectors
Flow Field

zonal velocity (nonlinear color bar)
Helium Model

Helium Equation

\[ \frac{\partial T}{\partial t} + \vec{u} \cdot \nabla T = \frac{\nabla \cdot (\alpha h \nabla T)}{h} \]

Helium Boundary Condition

<table>
<thead>
<tr>
<th>Boundary segment</th>
<th>Boundary Condition</th>
<th>( T ) or ( \nabla T \cdot \vec{n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Northeast (-10 \leq \theta)</td>
<td>Dirichlet</td>
<td>( \frac{3}{4} )</td>
</tr>
<tr>
<td>East ( \theta \leq -10)</td>
<td>Dirichlet</td>
<td>0</td>
</tr>
<tr>
<td>South ( \theta \approx -30)</td>
<td>Dirichlet</td>
<td>0</td>
</tr>
<tr>
<td>Inlet</td>
<td>Dirichlet</td>
<td>0</td>
</tr>
<tr>
<td>North</td>
<td>Neumann</td>
<td>0</td>
</tr>
<tr>
<td>West</td>
<td>Dirichlet</td>
<td>( T_b(\theta) )</td>
</tr>
</tbody>
</table>

Parameters \( \alpha \): 60 m\(^2\)/s.
Helium On Western Boundary

\[ T_b = \left\{ \frac{1}{2} \left[ 1 + \tanh \left( \frac{3}{4} (\theta + 30) \right) \right] + \frac{1}{4} \left[ 1 - \tanh(\theta + 15) \right] - \frac{1}{2} \right\} \left[ 1 - e^{-2(\theta - \theta_s)} \right] \]
Conclusions so far

- Ridge system critical to formation of zonal jets.
- Southeastern trend of $^3$He reproduced and suggests advective spreading
- Multi-layer simulation needed to increase realism of simulation
Tides in Hudson-Raritan Estuary
2D-Tidal circulation in New York/Newark bays
2D-Tidal circulation in New York/Newark bays
2D-Tidal circulation in New York/Newark bays

- Solve the shallow water equations
- Forcing from tides, wind, and river discharges
- Quadratic bottom drag to model bottom friction.
Comparison of Tidal Elevation in Bridgeport CT
Simulated flow patterns at high water

- 1.0 m/s
Simulated flow patterns 2 hours after high water

- 1.0 m/s
Simulated flow patterns at low water
Simulated flow patterns 2 hours after low water
Residual Circulation
Residual Circulation
Equations: Hydrostatic Boussinesq NS

\[
\begin{align*}
\frac{D\vec{u}}{Dt} + \mathbf{f} \times \vec{u} + g \nabla \zeta + \nabla p &= \nabla \cdot (\alpha \nabla \vec{u}) + \frac{\partial}{\partial z} \left( \nu \frac{\partial \vec{u}}{\partial z} \right) \\
\frac{\partial p}{\partial z} &= -g \frac{\rho}{\rho_0} \\
\nabla \cdot \vec{u} + \frac{\partial w}{\partial z} &= 0 \\
\frac{D}{Dt} \begin{pmatrix} T \\ S \end{pmatrix} &= \nabla \cdot \left[ \alpha_t \nabla \begin{pmatrix} T \\ S \end{pmatrix} \right] + \frac{\partial}{\partial z} \left[ \nu_t \frac{\partial \begin{pmatrix} T \\ S \end{pmatrix}}{\partial z} \right] \\
\rho &= \rho(T, S).
\end{align*}
\]
Peculiarities of Hydrostaticity

• Simple definition of pressure
  – Pressure computed from momentum equation
  – Column-wise vertical integration of density
  – Avoid expensive 3D elliptic solve

• $w$ is diagnostic
  – computed from continuity equation
  – must integrate horizontal divergence vertically
Depth-Averaged Equations

\[
\frac{\partial \zeta}{\partial t} + \nabla \cdot [(h + \zeta) U] = 0,
\]

\[
\frac{DU}{Dt} + f \times U + g \nabla \zeta = \vec{F} - \vec{C}
\]

\[
U = \frac{1}{h + \zeta} \int_{-h}^{\zeta} \vec{u} \, dz,
\]

\[
\vec{C} = \frac{1}{h + \zeta} \nabla \cdot \left[ \int_{-h}^{\zeta} (\vec{u} - U)(\vec{u} - U) \, dz \right] + \frac{1}{h + \zeta} \int_{-h}^{\zeta} \nabla p \, dz,
\]
SEOM-3D Highlights

- Terrain-Following Hexahedral elements
- Computational lines aligned with vertical
- Variables are co-located (A-Grid configuration)
- SSH staggered for spurious modes
DGM in SEOM-3D

- SSH and $T/S$ formulated in DGM
- $T/S$ (interior) Gauss roots (3D-staggering)
- Momentum equations solved with CGM
Gravitational Adjustment SEOM-3D

Domain: 64 km × 20 m
Resolution: 0.5 km × 1 m (e.g. 16-4km element N=8.)
Initial Density contrast across front: 5 kg/m³
## Gravitational Adjustment SEOM-3D

<table>
<thead>
<tr>
<th>diff</th>
<th>visc=50 m²/s</th>
<th></th>
<th>visc=75 m²/s</th>
<th></th>
<th>visc=100 m²/s</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CGM</td>
<td>DGM</td>
<td>CGM</td>
<td>DGM</td>
<td>CGM</td>
<td>DGM</td>
</tr>
<tr>
<td></td>
<td>ρ_{min}</td>
<td>ρ_{max}</td>
<td>ρ_{min}</td>
<td>ρ_{max}</td>
<td>ρ_{min}</td>
<td>ρ_{max}</td>
</tr>
<tr>
<td>0</td>
<td>-1.61</td>
<td>6.79</td>
<td>-1.15</td>
<td>6.07</td>
<td>-0.82</td>
<td>5.7</td>
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<tr>
<td>5</td>
<td>-1.50</td>
<td>6.59</td>
<td>-1.16</td>
<td>6.04</td>
<td>-3.84</td>
<td>39.97</td>
</tr>
<tr>
<td>10</td>
<td>-1.36</td>
<td>6.39</td>
<td>-3.76</td>
<td>40.96</td>
<td>-1.11</td>
<td>6.01</td>
</tr>
<tr>
<td>25</td>
<td>-7.05</td>
<td>52.3</td>
<td>-0.63</td>
<td>5.71</td>
<td>-1.21</td>
<td>10.37</td>
</tr>
<tr>
<td>50</td>
<td>-0.55</td>
<td>6.99</td>
<td>-0.32</td>
<td>5.05</td>
<td>-0.66</td>
<td>5.72</td>
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<tr>
<td>75</td>
<td>-0.12</td>
<td>5.32</td>
<td>-0.06</td>
<td>5.05</td>
<td>-0.12</td>
<td>5.18</td>
</tr>
<tr>
<td>100</td>
<td>-0.02</td>
<td>5.04</td>
<td>-0.03</td>
<td>5.04</td>
<td>-0.004</td>
<td>5.004</td>
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<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Overflow: CGM versus DGM

\[ \begin{align*}
\rho_2 \\
\rho_1 \\
\end{align*} \]

\[ t = 0 \quad \text{vs.} \quad t > 0 \]
Overflow: Spectral Element Grid

Domain: width = 200 km
200 m \leq depth \leq 4000 m
Grid: 40 \times 5 elements
CGM/DGM Velocity Int: 5 \times 3
CGM Tracer Int: 5 \times 3
DGM Tracer Int: 3 \times 2
Overflow: CGM versus DGM $\alpha = 100 \text{ m/s}^2$
Overflow: CGM versus DGM

Extrema of density in sigma units $0 \leq \rho \leq 5$
Viscosity is 1000 m$^2$/s
Viscosity and Diffusion along computational lines.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>CGM $\min \rho$</th>
<th>CGM $\max \rho$</th>
<th>DGM $\min \rho$</th>
<th>DGM $\max \rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2.41</td>
<td>5.63</td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>-1.54</td>
<td>5.35</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>-1.59</td>
<td>5.42</td>
<td>-1.50</td>
<td>5.22</td>
</tr>
<tr>
<td>500</td>
<td>-0.69</td>
<td>5.01</td>
<td>-1.37</td>
<td>5.20</td>
</tr>
<tr>
<td>1000</td>
<td>-0.61</td>
<td>5.00</td>
<td>-1.16</td>
<td>5.19</td>
</tr>
</tbody>
</table>
Non-Hydrostatic Simulation of Gravitational Adjustment

- Scale of mixing in ocean too small to be represented in current ocean models.

- Mixing is nevertheless important to produce observed water masses

  - e.g. Mediterranean outflow at Gibraltar strait.

- Mixing must be parametrized in OGCM.

- Need reference solutions to check mixing parametrization.
Non-Hydrostatic Simulation of Gravitational Adjustment

• Set-up similar to previous problem but...

• Geometry aspect ratio is 5:1 instead of 3200:1

• Equal resolution in vertical and horizontal to capture Kelvin-Helmoltz instabilities.
Dynamical Equations

- Solve vorticity-streamfunction equations

\[ \zeta_t + \vec{u} \cdot \zeta = -g \rho_x + \nabla \cdot \nu \nabla \zeta + \nabla \times \mathbf{F} \]

\[ \nabla^2 \psi + \zeta = 0, \quad \vec{u} = (\psi_y, -\psi_x) \]

\[ \rho_t + \vec{u} \cdot \rho = \nabla \cdot \alpha \nabla \rho \]

- CGM for streamfunction velocity solution

- DGM for density advection
Non-Hydrostatic Simulation of Gravitational Adjustment

- 128 × 25 elements with 15×15 points

- $Re = 10,000$, $Pr = \frac{\nu}{\alpha} = 7$

- $\Delta t = 4 \times 10^{-8}$

- 5 processor simulation
Conclusion

- Ocean Simulation with FEM promising

- DGM has improved model’s robustness.

- Challenges remain for 3D models
  - Optimal vertical representation (ALE?)
  - Grid generation to resolve steep topography
  - Parallel iterative solvers designed for thin domains
– How to manage complex models.

– How to build a community FE model
THANK YOU